Action of a Frobenius-like group

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Abstract

We call a finite group Frobenius-like if it has a nontrivial nilpotent normal subgroup F possessing a nontrivial complement H such that $[F,h]=F$ for all nonidentity elements $h \in H$. We prove that any irreducible nontrivial FH-module for a Frobenius-like group FH of odd order over an algebraically closed field has an H-regular direct summand if either F is fixed point free on V or F acts nontrivially on V and the characteristic of the field is coprime to the order of F. Some consequences of this result are also derived.

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Theorem A. *Let $G \lhd GA$ such that A normalizes a Sylow system of G. Suppose that $G' \neq G$ and $[G,a]=G$ for all nonidentity elements $a \in A$. Let V be a nonzero vector space over an algebraically closed field k and let GA act on V as a group of linear transformations such that char(k) does not divide the order of A. Then V_A has a proper A-regular direct summand if one of the following holds:*

(i) $C_V(G)=0$,

(ii) $[V,G] \neq 0$ and char(k) does not divide the order of G.

The proof of this theorem relies on the following result which can be regarded

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as a generalization of [[2], V.17.13], and is of independent interest too.

Theorem B. *Let H be a group in which each Sylow subgroup is cyclic. Assume that $H/F(H)$ is not a nontrivial 2-group. Let P be an extraspecial group of order p^{2m+1} for some prime p not dividing $|H|$. Suppose that H acts on P in such a way that H centralizes $Z(P)$, and $[P,h]=P$ for any nonidentity element $h \in H$. Let k be an algebraically closed field of characteristic not dividing the order of $G=PH$ and let V be a kG-module on which $Z(P)$ acts nontrivially and P acts irreducibly. Let χ be the character of G afforded by V. Then $|H|$ divides $p^m-\delta$ and $\chi_H=\frac{p^m-\delta}{|H|}\rho+\delta\mu$ where ρ is the regular character of H, μ is a linear character of H and $\delta \in \{-1,1\}$. In particular, V_H contains the regular kH-module as a direct summand if G is of odd order.*

It should be noted that if G is not of odd order, then the module V_H need not to contain the regular kH-module.

We want to draw the attention of the reader to Theorem 3.2 and Theorem 3.4 in the remarkable paper [8] of Turull which are very close to Theorem B and Theorem A respectively.

As applications of Theorem A and Theorem B we obtain the following:

Corollary C. *Let G be a finite solvable group acted on coprimely by a Frobenius-like group FH of odd order so that $[G,F]\neq 1$. Then $C_G(H)\neq 1$.*

Corollary D. *Let P be a p-group acted on coprimely by a Frobenius-like group FH of odd order so that $[P,F]=P$. Then*
(i) the nilpotency class of P is at most $2log_p|C_P(H)|$,
(ii) $|P|$ is bounded in terms of $|F|$ and $|C_P(H)|$,
(iii) the rank of P is bounded in terms of $|F|$ and the rank of $C_P(H)$.

In the present paper all groups are assumed to be finite. The notation and terminology are standard, and the *rank* of a finite group is the minimum number r such that every subgroup is generated by r elements.

1. Existence of regular modules

In this section we prove a technical result pertaining to the main result of this paper, which can be regarded as a generalization of [2, V.17.13]. We begin with a preliminary lemma.

Lemma 1.1. *Let FH be a group with $F \lhd FH$, $F' \neq F$ and $[F,h] = F$ for all nonidentity elements $h \in H$. Assume that all Sylow subgroups of H are cyclic. Then*
(i) the groups H' and H/H' are cyclic of coprime orders,
(ii) $H = H'\langle y\rangle = H_0\langle y\rangle$ with $H' \cap \langle y\rangle = 1$ for some $y \in H$ where H_0 denotes the Fitting subgroup of H, and $H_0 = H' \times C_{\langle y\rangle}(H')$ is cyclic,
(iii) $\pi(H_0) = \pi(H)$.

Proof. The group FH/F' is Frobenius with Frobenius complement isomorphic to H. Then *(i)* follows by [[3], Theorem 5.16]. In particular, $H = H'\langle y\rangle$ for some $y \in H$ with $H' \cap \langle y\rangle = 1$. On the other hand the group H has a unique subgroup of order p for each prime p dividing its order by the argument applied in the proof of Theorem 6.19 in [3] which relies on [[3], Theorem 6.9]. Hence $\pi(H_0) = \pi(H)$ as claimed in *(iii)*. Let now H_0 denote the Fitting subgroup of H. Then $H_0 = H'(H_0 \cap \langle y\rangle)$ and $[H_0 \cap \langle y\rangle, H'] = 1$, that is, $H_0 \cap \langle y\rangle \subseteq C_{\langle y\rangle}(H') \subseteq H_0$. This establishes the claim *(ii)*. $\square$

Theorem B. *Let H be a group in which each Sylow subgroup is cyclic. Assume that $H/F(H)$ is not a nontrivial 2-group. Let P be an extraspecial group of order p^{2m+1} for some prime p not dividing $|H|$. Suppose that H acts on P in such a way that H centralizes $Z(P)$, and $[P,h] = P$ for any nonidentity element $h \in H$. Let k be an algebraically closed field of characteristic not dividing the order of $G = PH$ and let V be a kG-module on which $Z(P)$ acts nontrivially and P acts irreducibly. Let χ be the character of G afforded by V. Then $|H|$ divides $p^m - \delta$ and $\chi_H = \frac{p^m-\delta}{|H|}\rho + \delta\mu$ where ρ is the regular character of H, μ is a linear character of H and $\delta \in \{-1, 1\}$. In particular, V_H contains the regular kH-module as a direct summand if G is of odd order.*

Proof. Since all Sylow subgroups of H are cyclic and $G/Z(P)$ is a Frobenius group with a complement isomorphic to H, we see that H has the properties described in Lemma 2.1. By [[2], V.17.13] we can assume that H is not nilpotent and recall that $H/F(H)$ is not a 2-group by hypothesis.

Note that $dimV = p^m$ as χ_P is a faithful irreducible character of P. Let D be the representation of G afforded by the module V and let M be the k-space of square matrices of size p^m over k. We define a left kH-module structure on M by letting

$$h \cdot X := D(h)XD(h^{-1}), \text{ for any } X \in M \text{ and for any } h \in H.$$

It is known that H acts on $Hom_k(V,V)$ via the multiplication $(h \cdot T)(v) = hT(h^{-1}v)$ for any $h \in H$, $T \in Hom_k(V,V)$, and $v \in V$. Then clearly M is isomorphic to the $k[H]$-module $Hom_k(V,V)$. Furthermore $Hom_k(V,V)$ and $V^* \otimes V$ are isomorphic as $k[H]$-modules. So by letting $Irr(H) = \{\psi_1, \psi_2, \ldots, \psi_s\}$ and $\chi_H = \sum_{i=1}^{s} n_i\psi_i$ with nonnegative integers $n_i, i = 1, \ldots s$, we have $\Psi = \sum_{k,l=1,\ldots,s} n_k n_l \overline{\psi_k}\psi_l$ where Ψ is the character of H afforded by M.

Choose a transversal T for $Z(P)$ in P. Then the set $\{D(x)|x \in T\}$ forms a basis for M by a result of Burnside [[2], V.5.14] and the fact that $D(zx) = \lambda(z)D(x)$ for any $x \in T$ and $z \in Z(P)$. Notice that $P/Z(P)$ is the union of one H-orbit of length 1 and $d = \frac{p^{2m}-1}{|H|}$ orbits of length $|H|$. Thus we have $M = \langle I \rangle \oplus M_1 \oplus \cdots \oplus M_d$ with $M_i \cong k[H]$ as H-module for any $i = 1, 2, \ldots, d$. So we get

$$\Psi = 1_H + \sum_{i=1}^{s} \tfrac{p^{2m}-1}{|H|}\psi_i(1)\psi_i = \sum_{k,l=1}^{s} n_k n_l \overline{\psi_k}\psi_l.$$

Thus the multiplicity of the principal character 1_H in Ψ is

$$[1_H, \Psi]_H = 1 + \tfrac{p^{2m}-1}{|H|} = \sum_{k=1}^{s} n_k^2$$

and the multiplicity of any nonprincipal $\alpha \in Irr(H)$ in Ψ is

$$[\alpha, \Psi]_H = \tfrac{p^{2m}-1}{|H|}\alpha(1) = \sum_{k,l=1}^{s} n_k n_l(\psi_l, \psi_k\alpha).$$

In particular for any nonprincipal linear character γ of H we have

$$\tfrac{p^{2m}-1}{|H|} = \textstyle\sum_{\alpha \in Irr(H)} n_\alpha n_{\alpha\gamma}.$$

This gives

$$1 = \textstyle\sum_{\alpha \in Irr(H)} n_\alpha^2 - \sum_{\alpha \in Irr(H)} n_\alpha n_{\alpha\gamma}, \text{ and hence } 2 = \sum_{\alpha \in Irr(H)} (n_\alpha - n_{\alpha\gamma})^2$$

for any nonprincipal linear character γ of H.

The group $\widehat{H/H'}$ of characters of the abelian group H/H' is isomorphic to H/H'. In particular it is cyclic. Let ϑ be a generator of $\widehat{H/H'}$. It acts on *Irr(H)* by multiplication. Let Φ_i, $i = 1, \ldots, b$ be the orbits of ϑ on *Irr*(H) and let $m_i = |\Phi_i|$. Then we have $2 = \sum_{i=1}^{b} \sum_{\alpha \in \Phi_i} (n_\alpha - n_{\alpha\vartheta})^2$. So there are exactly two elements β and γ in *Irr*(H) such that $|n_\beta - n_{\beta\vartheta}| = 1 = |n_\gamma - n_{\gamma\vartheta}|$, and we have $n_\alpha = n_{\alpha\vartheta}$ for any $\alpha \in$ *Irr*$(H) - \{\beta, \gamma\}$. If $\beta \in \Phi_i$ and $\gamma \notin \Phi_i$, then $n_\beta \neq n_{\beta\vartheta} = n_{\beta\vartheta^2} = \cdots = n_{\beta\vartheta^{m_i - 1}} = n_\beta$, which is not possible. So if necessary by reindexing the orbits, we can assume that β and γ are both elements of Φ_b, and $n_\alpha = n_{\alpha\vartheta}$ for any $i = 1, 2, \ldots, b-1$ and any $\alpha \in \Phi_i$.

Suppose that $\gamma = \beta\vartheta^u$ for some $u \in \{1, 2, \ldots, m_b - 1\}$. We have

$$n_\beta \neq n_{\beta\vartheta} = \cdots = n_{\beta\vartheta^u} \neq n_{\beta\vartheta^{u+1}} = \cdots = n_{\beta\vartheta^{m_b - 1}} = n_\beta.$$

Since each Φ_i is either a ϑ^2-orbit or the union of two ϑ^2-orbits of the same size we get

$$2 = \sum_{i=1}^{b} \sum_{\alpha \in \Phi_i} (n_\alpha - n_{\alpha\vartheta^2})^2 = \sum_{\alpha \in \Phi_b} (n_\alpha - n_{\alpha\vartheta^2})^2.$$

So the differences $n_\beta - n_{\beta\vartheta^2}, n_{\beta\vartheta^{m_b-1}} - n_{\beta\vartheta}$, $n_\gamma - n_{\gamma\vartheta^2}$ are all nonzero if $u \in \{2, \ldots, m_b - 2\}$, which is a contradiction. If necessary by replacing ϑ by ϑ^{-1} we can assume that $n_\beta \neq n_{\beta\vartheta} \neq n_{\beta\vartheta^2} = \cdots = n_{\beta\vartheta^{m_b-1}} = n_{\beta.}$ We let $n_{\beta\vartheta} = n_\beta + \delta$, with some $\delta \in \{-1, 1\}$. Choose an element α_i from $\Phi_i, i = 1, 2, \ldots, b-1$, and let $\alpha_b = \beta$. Then

$$\chi_{_H} = \sum_{i=1}^{b} n_{\alpha_i} (\alpha_i + \alpha_i\vartheta + \cdots \alpha_i\vartheta^{m_i - 1}) + \delta\mu, \text{ where } \mu = \alpha_b\vartheta.$$

By [[2], V.17.13] we have

$$\chi_{_{H'}} = \frac{p^m - \delta'}{|H'|}\rho' + \delta'\mu' = (\sum_{i=1}^{b} n_{\alpha_i} m_i \alpha_i)_{_{H'}} + \delta\alpha_{b_{_{H'}}}$$

for some $\delta' \in \{-1, 1\}$ and $\mu' \in Irr(H')$ where ρ' is the regular character of H'.

It follows by [[4], Exercise 6.2] that if $i \neq j$ then the sets of irreducible constituents of the restrictions of α_i and α_j are disjoint. By Clifford's theorem we have

$\alpha_{i\,H'} = e_i \sum_{j=1}^{t_i} \lambda_{i,j}$ where $I_H(\lambda_{i,1}) = T_i,\ t_i = [H : T_i],\ H = \bigcup_{j=1}^{t_i} T_i x_{i,j}$, and $\lambda_{i,j} = \lambda_{i,1}^{x_{i,j}}, j = 1, 2, \ldots, t_i; i = 1, 2, \ldots, b$. Now $\{\lambda_{i,j} | j = 1, 2, \ldots, t_i; i = 1, 2, \ldots, b\} = Irr(H')$.

It is known that there is a unique $\xi_i \in Irr(T_i)$ such that ${\xi_i}^H = \alpha_i$ and ${\xi_i}_{_{H'}} = e_i\lambda_{i,1}$. On the other hand as T_i/H' is cyclic, $\lambda_{i,1}$ has an extension, say φ, to T_i. But then φ^H must belong to the ϑ-orbit containing α_i which implies ${\alpha_i}_{_{H'}} = (\varphi^H)_{_{H'}}$. Therefore we have

$$e_i = [{\alpha_i}_{_{H'}}, \lambda_{i,1}] = [(\varphi^H)_{_{H'}}, \lambda_{i,1}] = [\varphi_{_{H'}}, \lambda_{i,1}] = 1 \text{ for any } i = 1, 2, \ldots, b.$$

Let now $e = \frac{p^m - \delta'}{|H'|}$ and $\mu' = \lambda_{i_0,j_0}$. Then for any $\upsilon \in \widehat{H'}$ we have

$$[\chi_{_{H'}}, \upsilon]_{_{H'}} = \begin{cases} e & \text{if } \quad \upsilon \neq \mu' \\ e + \delta' & \text{if } \quad \upsilon = \mu' \end{cases}.$$

Set $H_0 = F(H)$. Applying [[2], V.17.13] to the action of PH_0 on V we see in particular that $|H_0|$ divides $p^m - \delta^*$ for some $\delta^* \in \{-1, 1\}$. Then $|H'|$ divides $\delta - \delta^* = (p^m - \delta^*) - (p^m - \delta)$ and so we have either $\delta = \delta^*$ or $|H'| = 2$. If the latter holds then $H' \leq Z(H)$ and hence H is abelian, which is not the case. Thus $|H_0/H'|$ divides e. In particular $e > 1$ and so $e + \delta' > 0$ which shows that $[\chi_{_H}, \alpha_{i_0}]_{_H} \neq 0$.

If $t_{i_0} \neq 1$, then there exists $j_1 \neq j_0$ such that

$$e = [\chi_{_{H'}}, \lambda_{i_0,j_1}]_{_{H'}} = [\chi_{_{H'}}, \lambda_{i_0,j_0}]_{_{H'}} = e + \delta'$$

which is not possible. Then $t_{i_0} = 1$ and hence μ' is H-invariant. This yields that ${\alpha_{i_0}}_{_{H'}} = \mu' = \lambda_{i_0,1}$. In particular α_{i_0} is a linear character of H and so $m_{i_0} = |H/H'|$.

Furthermore we have

$$e+\delta' = \begin{cases} n_{\alpha_{i_0}} m_{i_0} & \text{if} \quad i_0 < b \\ n_{\alpha_b} m_b + \delta & \text{if} \quad i_0 = b \end{cases} .$$

Now $|H_0/H'|$ divides the greatest common divisor of e and m_{i_0} which forces that $i_0 = b$ as H_0/H' is nontrivial. Furthermore if $\delta \neq \delta'$ we have $|H_0/H'| = 2$, which implies by Lemma 2.1 that H/H' is a 2-group. This contradiction shows that $\delta = \delta'$ and hence $n_{\alpha_b} m_b = e$ by the above formula. In particular $\frac{p^m-\delta}{|H|} = n_{\alpha_b}$ is an integer. On the other hand we also have $e = [\chi_{_{H'}}, \lambda_{i,1}]_{H'} = n_{\alpha_i} m_i$ if $i < b$.

Set next $r_i = |T_i/H'|$. As $\widehat{T_i/H'} = \langle \vartheta|_{T_i} \rangle$ we obtain $T_i \leq Ker\vartheta^{r_i}$. As $\alpha_i = {\xi_i}^H$ for some ξ_i of T_i and T_i is normal in H, we observe that $\alpha_i(x) = 0$ for any $x \notin T_i$. Combining these two observations we get $\vartheta^{r_i}\alpha_i = \alpha_i$. Thus m_i divides r_i and hence $|H/H'| = r_i t_i = m_i c_i t_i$ for some positive integer c_i. It follows now that $n_{\alpha_i} m_i c_i t_i = e c_i \alpha_i(1)$ and hence $n_{\alpha_i} = \frac{p^m-\delta}{|H|} c_i \alpha_i(1) \geq \frac{p^m-\delta}{|H|} \alpha_i(1)$. Thus $\frac{p^m-\delta}{|H|}\rho + \delta\mu$ occurs in $\chi_{_H}$. As the degrees of these characters are the same we see that they are equal. This completes the proof of the theorem. $\square$

The next example shows that the hypothesis about the structure of H can not be avoided.

Example. Let V be the $GF(3)$-space $GF(3^4)$. We define the map $(\cdot|\cdot) : V \times V \longrightarrow GF(3)$ by $(\cdot|\cdot)(x,y) = Tr(d \cdot (xy^9 - x^9 y))$ for $x, y \in V$, where d is an element of order 16 in $GF(3^4)^*$. One can check that $(\cdot|\cdot)$ is a nonsingular symplectic form on V.

Let $b \in GF(3^4)^*$ be an element of order 5 and $c \in GF(3^4)^*$ be an element of order 4. We define $\tau_b : V \longrightarrow V$ by $\tau_b(x) = b \cdot x$ and $\sigma : V \longrightarrow V$ by $\sigma(x) = c \cdot x^9$. Then $H = \langle \tau_b, \sigma \rangle$ is a subgroup of $GL(4,3)$ preserving the symplectic form, with $|H| = 20, H' = \langle \tau_b \rangle$ of order 5, and $F(H) = H' \times \langle \sigma^2 \rangle$ of order 10. Furthermore $h(v) = v$ for some $0 \neq v \in V$ and $h \in H$ implies that $h = 1$. So if P is the extraspecial group of order 3^5 and exponent 3, then it admits H as a subgroup of automorphisms of P, centralizing $Z(P)$ and satisfying $[P,h] = P$ for any nonidentity element $h \in H$. Let χ be any irreducible character of the group PH which does not contain $Z(P)$ in its kernel. Clearly, we have $\chi_{_H} \neq \frac{3^2-\delta}{|H|}\rho + \delta\mu$ for the regular H-character ρ and any $\delta \in \{-1, 1\}$ and $\mu \in Irr(H)$, because $\frac{3^2-\delta}{|H|}$ is not an integer.

2. Action of a Frobenius-like group

We define a slight generalization of Frobenius groups which we call Frobenius-like groups and prove the main result of this paper.

Definition 2.1. *Let F and H be nontrivial finite groups such that H acts on F via automorphisms. Assume that F is nilpotent and $[F, h] = F$ for all non-identity elements $h \in H$. We call the semidirect product FH a "Frobenius-like group" with kernel F and complement H.*

Lemma 2.2. *Let FH be a group with $F \lhd FH$, and $[F, h] = F$ for all nonidentity elements $h \in H$. Let FH act on the set X. If F acts nontrivially on X then H acts faithfully on X.*

Proof. Let K denote the kernel of FH on X. If $K \cap H \neq 1$ then we have

$$F = [F, K \cap H] \leq K.$$

This contradiction proves the claim. $\Box$

Theorem A. *Let V be a nonzero vector space over an algebraically closed field k and let FH be a Frobenius-like group of odd order acting on V as a group of linear transformations such that $char(k)$ does not divide the order of H. Then V_H has a proper H-regular direct summand if one of the following holds:*

(i) $C_V(F) = 0$,

(ii) $[V, F] \neq 0$ and $char(k)$ does not divide the order of F.

Proof. Assume that the theorem is false and choose a counter-example with minimum $dimV + |FH|$. We shall proceed in several steps.

(1) We may assume that $char(k)$ does not divide the order of F and F is a q-group for some prime q with $C_V(F) = 0$.

Set $char(k) = p$. As F is nilpotent, we have $F = F_p \times F_{p'}$. If *(i)* holds then $C_V(F_{p'}) = 0$. Notice also that $[F_{p'}, h] = F_{p'}$ for every nonidentity element

$h \in H$. So by an induction argument applied to the action of $F_{p'}H$ on V we see that F is a p'-group.

Let now q be a prime dividing the order of F such that $[V, F_q] \neq 0$. As the action of F_qH on $[V, F_q]$ satisfies the hypothesis of the theorem it follows by induction that $V = [V, F_q]$ and $F = F_q$.

(2) The group FH acts irreducibly and faithfully on V.

Let U be an irreducible FH submodule of V. Note that $C_U(F) \subseteq C_V(F) = 0$. It follows now by induction that U has a proper H-regular direct summand and hence so does V. Therefore V is an irreducible FH-module as claimed.

Notice next that $C_{FH}(V) = C_F(V)C_H(V)$. As a consequence of Lemma 2.2 we have $C_H(V) = 1$. Now an induction argument applied to the action of $(F/C_F(V))H$ on V yields that $C_F(V) = 1$ which completes the proof of the claim.

(3) V_F is homogeneous and hence F is nonabelian.

By Clifford's theorem the module V is a direct sum of homogeneous F-modules permuted transitively by H. We pick now an F-homogeneous component W of V and set $H_1 = Stab_H(W)$. If $H_1 = 1$, then V is free as a kH-module obviously. Thus we may assume that $H_1 \neq 1$. Applying induction to the action of FH_1 on W we conclude that W has a proper H_1-regular direct summand and hence V has a proper H-regular direct summand, as desired. This forces now that $H_1 = H$, that is, V_F is homogeneous.

Assume next that F is abelian. Then F acts by scalars on V and so we have $F = [F, h] = 1$ for every $h \in H$ which is not the case. Therefore F is a nonabelian group as claimed.

(4) V_F is irreducible.

By *(3)*, $V \cong X \oplus \cdots \oplus X$ for some irreducible kF-module X. Note also that for every $h \in H$ $V^h = V$ and hence X^h and X are isomorphic as kF-modules. As H acts coprimely on F, Corollary 8.16 in [4] yields that the module X can be extended to an FH-module Y in a unique way subject to the condition that if $x \in H$ then $det_Y(x) = 1$. Then by Corollary 6.17 in [4] there is a $k(FH/F)$-module U where $V \cong Y \otimes U$. It should be noted that H acts faithfully on

Y by Lemma 2.2. An induction argument applied to the action of the group $(F/Ker(FonY))H$ on Y shows that F acts faithfully on Y and also $dim_k V = dim_k Y$. Thus $dim_k U = 1$ and hence $X = V_F$ establishing the claim.

(5) V_M is homogeneous for every maximal FH-invariant subgroup M of F.

Pick a maximal FH-invariant subgroup M of F. Then F/M is an elementary abelian group on which H acts irreducibly. Furthermore, the group $(F/M)H$ is Frobenius as $C_{F/M}(h) = 1$ for every nonidentity element $h \in H$. By *(4)*, V_F is irreducible. By Clifford's theorem there is a collection $\{U_1, \ldots, U_s\}$ of homogeneous M-modules permuted transitively by F such that $V_M = \bigoplus_{i=1}^{s} U_i$. On the other hand $M \lhd FH$ and the components $U_1, \ldots, U_s$ are permuted transitively also by FH. Then by setting $F_0 = Stab_F(U_1)$, we see that $s = |F : F_0| = |FH : Stab_{FH}(U_1)|$. It follows now that $s = \frac{|F||H|}{|Stab_{FH}(U_1)|} = \frac{|F|}{|F_0|}$ whence

$|Stab_{FH}(U_1)| = |F_0|\,|H|$. As $(|F_0|\,, |Stab_{FH}(U_1)/F_0|) = 1$, a complement H_0 of F_0 in $Stab_{FH}(U_1)$ exists. Therefore without loss of generality we may assume that $H \leq Stab_{FH}(U_1)$, that is, $Stab_{FH}(U_1) = F_0 H$.

On the other hand, F_0/M is either trivial or equal to F/M due to the irreducible action of H on F/M. If trivial, then F/M acts regularly on $\{U_1, \ldots, U_s\}$. So for all $i = 1, \ldots, s$, there is a unique $\bar{x_i} \in F/M$ such that $U_i = U_1^{\bar{x_i}}$. Then we have $U_1^{\bar{x_i}h} = U_1^{\bar{x_i}^h}$ for all $h \in H$. This means that H acts regularly on $\{U_2, \ldots, U_s\}$ and hence for any $0 \neq w \in U_2$, the set $\left\{w^h | h \in H\right\}$ forms a basis for a free kH-module. Thus we may assume that $F_0 = F$. In particular U_1 is FH-invariant and hence V_M is homogeneous as claimed.

(6) V_S is homogeneous for every FH-invariant subgroup S of F.

Let S be an FH-invariant subgroup of F. Now by *(4)* V_F is irreducible. Then, by Clifford's theorem we have $V_S = \bigoplus_{i=1}^{s} U_i$ for a collection of S-homogeneous modules $\{U_1, \ldots, U_s\}$ permuted transitively by F. On the other hand $S \lhd FH$ and the components $U_1, \ldots, U_s$ are permuted transitively also by FH. We set now $F_0 = Stab_F(U_1)$. Then $s = |F : F_0| = |FH : Stab_{FH}(U_1)|$ and hence we may assume by a similar argument as in the proof of claim *(5)* that H stabilizes U_1.

If $s \neq 1$, F_0 is contained in a maximal subgroup, say K, of F. However, every maximal subgroup and hence K is normal in F as F is nilpotent. In fact F_0 is H-invariant and hence $F_0 \leq \cap_{h \in H} K^h \lhd FH$. Now $\cap_{h \in H} K^h$ is contained in a maximal FH-invariant subgroup, say M, of F. It follows then by *(5)* that V_M is homogeneous, that is, $V \cong X \oplus \cdots \oplus X$ for some irreducible M-module X. We consider the decomposition of X into its S-homogeneous components; more precisely we have $X_S = Y_1 \oplus \cdots \oplus Y_r$ for S-homogeneous modules $Y_1, \ldots, Y_r$ by Clifford's theorem. Clearly $r = s$ and $Y_i = X \cap Y$ for each $i = 1, \ldots, s$. Since M acts transitively on the set $\{Y_1, \ldots, Y_s\}$, its action on the set $\{U_1, \ldots, U_s\}$ has to be transitive also. So $|M : Stab_M(U_1)| = s = |F : F_0|$. As $F_0 \leq M$ we have the equality $F = M$, which is a contradiction. Therefore $s = 1$, that is, V_S is homogeneous as claimed.

(7) F is extraspecial such that $Z(F) \leq Z(FH)$, and the theorem follows.

Pick a characteristic abelian subgroup S of F. By the above claim V_S is homogeneous and hence S is cyclic. Applying [[2], page 360, Aufgabe 33] to the action of H on F we see that the group F is either cyclic or extraspecial. Recall that F is nonabelian by *(3)*. Then the group F is extraspecial as desired. As $V_{Z(F)}$ is homogeneous we also see that $Z(F) \leq Z(FH)$. Now Theorem B applied to the group FH on V shows that V_H contains the regular kH-module. This completes the proof of the theorem. $\square$

Remark. Notice that if FH is not of odd order, then the theorem above is not true due to the following observation:

For a prime p, let F be an extraspecial group of order p^{2m+1} and H be the cyclic group of order $p^m + 1$. There is a regular action of H on $F/\Phi(F)$ so that $[\Phi(F), H] = 1$. Therefore FH is Frobenius-like. By [[2], V.17.13] there exists an irreducible and faithful kFH-module V over an algebraically closed field of characteristic coprime to the order of FH such that $V_H \oplus U \cong kH$ where U is the irreducible trivial H-module. In particular V does not contain any submodule isomorphic to the regular H-module, more precisely $C_V(H) = 0$.

3. Applications

Corollary C. *Let G be a finite solvable group acted on coprimely by a Frobenius-like group FH of odd order so that $[G,F] \neq 1$. Then $C_G(H) \neq 1$.*

Proof. We proceed by induction on the order of G. Then F acts trivially on every proper FH-invariant subgroup of G and hence G is a q-group for some prime q. Theorem A applied to the action of FH on $V = G/\Phi(G)$ gives the result. □

Corollary D. *Let P be a p-group acted on coprimely by a Frobenius-like group FH of odd order so that $[P,F] = P$. Then*

(i) the nilpotency class of P is at most $2log_p|C_P(H)|$,

(ii) $|P|$ is bounded in terms of $|F|$ and $|C_P(H)|$,

(iii) the rank of P is bounded in terms of $|F|$ and the rank of $C_P(H)$.

Proof. (i) Notice that by theorem the group H fixes a point in each FH-invariant section of P on which F acts nontrivially. Then the proof goes similarly as in the proof of Theorem 1(a) in [6].

(ii) It suffices to bound $|P/P'|$ in the required form since the nilpotency class of P is bounded in terms of $|C_P(H)|$ by (i). We consider now a series

$$P_0 = P' \leq P_1 \leq P_2 \leq \ldots \leq P_m = P$$

of FH-invariant normal subgroups of P such that $E_i = P_i/P_{i-1}$ is an irreducible FH-module for each $i = 1, \ldots, m$. Due to coprime action of F on P and the fact $[P,F] = P$ we see that $C_{E_i}(F) = 0$. Then Theorem A applied to the action of FH on E_i yields $C_{E_i}(H) \neq 0$ for every $i = 1, \ldots, m$. As $dim\ E_i \leq |FH|$ we get $|E_i| \leq |C_{E_i}(H)|^{|FH|}$ and hence

$$|P/P'| = \textstyle\prod_{i=1}^m |E_i| \leq (\prod_{i=1}^m |C_{E_i}(H)|)^{|FH|} = (C_{P/P'}(H))^{|FH|}.$$

That is $|P/P'|$ is bounded in terms of $|F|$ and $|C_P(H)|$ since clearly $|H|$ is bounded in terms of $|F|$.

(iii) This can be proven in a similar fashion as in [6] with obvious changes that is using Corollary 4.1 instead of Lemma 1.2 in [6]. □

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